

Factors of almost squares and lattice points on circles

Tsz Ho Chan

Abstract

In this paper, we consider a conjecture of Erdős and Rosenfeld and a conjecture of Ruzsa when the number is an almost square. By the same method, we consider lattice points of a circle close to the x -axis with special radii.

1 Introduction and main result

In [3], Erdős and Rosenfeld considered the differences between the divisors of a positive integer n . They exhibited infinitely many integers with four “small” differences and posed the question that any positive integer can have at most a bounded number of “small” differences. Specifically, they asked

Conjecture 1 *Is there an absolute constant K , so that for every c , the number of divisors of n between $\sqrt{n} - c\sqrt[4]{n}$ and $\sqrt{n} + c\sqrt[4]{n}$ is at most K for $n > n_0(c)$?*

They also mentioned a conjecture of Ruzsa which is a stronger question:

Conjecture 2 *Given $\epsilon > 0$, is there a constant K_ϵ such that, for any positive integer n , the number of divisors of n between $n^{1/2} - n^{1/2-\epsilon}$ and $n^{1/2} + n^{1/2-\epsilon}$ is at most K_ϵ ?*

In [1], the author proved the above conjecture of Erdős and Rosenfeld [3] for perfect squares and made a tiny progress towards Ruzsa’s conjecture for perfect squares:

Theorem 1 *Any sufficiently large perfect square $n = N^2$ has at most five divisors between $\sqrt{n} - \sqrt[4]{n}(\log n)^{1/7}$ and $\sqrt{n} + \sqrt[4]{n}(\log n)^{1/7}$.*

Here we extend the result slightly to almost squares, namely

Theorem 2 *Any sufficiently large integer n , which can be factored as $(N - a)(N + b)$ for some integers N , a , b with $0 \leq a \leq b \leq e^{(\log n)^{2/7}}$, has at most eighteen divisors between $\sqrt{n} - \sqrt[4]{n}(\log n)^{1/14}$ and $\sqrt{n} + \sqrt[4]{n}(\log n)^{1/14}$.*

This includes numbers of the form $N^2 - 1$, $N^2 - 4$, $N^2 - N - 6$, ...

Factoring $n = xy$ can also be thought of as finding lattice points on the hyperbola $xy = n$. The above theorems mean that if $xy = n$ has two lattice points within a distance of $(\log n)^{1/7}$ from the point (\sqrt{n}, \sqrt{n}) , then it can have at most thirty six distinct lattice points within a distance $\sqrt[4]{n}(\log n)^{1/7}$ from the point (\sqrt{n}, \sqrt{n}) . Similarly, one can consider lattice points on the circle $x^2 + y^2 = n$. It was conjectured that (see [2] for example)

Conjecture 3 *For any $\alpha < 1/2$, there exists a constant C_α such that for any N we have*

$$\#\{(a, b) : a, b \text{ integers}, a^2 + b^2 = n, N \leq |b| < N + n^\alpha\} \leq C_\alpha.$$

A special case of interest is when $N = 0$:

$$\#\{(a, b) : a, b \text{ integers}, a^2 + b^2 = n, |b| < n^\alpha\} \leq C_\alpha. \quad (1)$$

It is simple to prove (1) for $\alpha \leq 1/4$. We extend the range for $|b|$ slightly in special cases of n by showing

Theorem 3 *For sufficiently large perfect squares $n = N^2$,*

$$\#\{(a, b) : a, b \text{ integers}, a^2 + b^2 = n, |b| < n^{1/4}(\log n)^{1/7}\} \leq 10.$$

Theorem 4 *For sufficiently large n , if $n = a_1^2 + b_1^2$ for some $|b_1| \leq e^{(\log n)^{2/7}}$, then*

$$\#\{(a, b) : a, b \text{ integers}, a^2 + b^2 = n, |b| < n^{1/4}(\log n)^{1/14}\} \leq 36.$$

2 Tools

The main tool of the proofs is the following result of Turk [4] on the size of solutions to simultaneous Pell equations.

Theorem 5 *Let a, b, c, d be squarefree positive integers with $a \neq b$ and $c \neq d$ and let e and f be any integers. If $af = ce$ then we also assume that $abcd$ is not a perfect square. Then every positive integer solution of*

$$\begin{cases} ax^2 - by^2 = e \\ cx^2 - dz^2 = f \end{cases}$$

satisfies

$$\max(x, y, z) < e^{C\alpha^2(\log \alpha)^3 \gamma \log \gamma}$$

where $\alpha = \max(a, b, c, d)$, $\beta = \max(|e|, |f|, 3)$, $\gamma = \max(\alpha \log \alpha, \log \beta)$ and C is a large absolute constant.

As a consequence, Turk [4] proved the following

Theorem 6 *Suppose $[N, N + K]$, where $K \geq 3$, contains three distinct integers of the form $a_i x_i^2$ with positive integers a_i, x_i for $i = 1, 2, 3$. Put $H = \max(a_1, a_2, a_3, 3)$. Then, for some absolute constant C ,*

$$CH^2(\log H)^3(H \log H + \log K)(\log H + \log \log K) > \log N.$$

We also need a result on almost squares.

Lemma 1 *Suppose a positive integer n can be factor as $n = (N - a_1)(N + b_1) = (N - a_2)(N + b_2)$ with $0 \leq a_1 < a_2$ and $0 \leq b_1 < b_2$. Then $a_2 b_2 \geq N$.*

Proof: From $(N - a_1)(N + b_1) = (N - a_2)(N + b_2)$, we get $(b_1 - a_1)N - a_1 b_1 = (b_2 - a_2)N - a_2 b_2$. Then $[(b_2 - a_2) - (b_1 - a_1)]N = a_2 b_2 - a_1 b_1 > 0$. Hence $N \leq [(b_2 - a_2) - (b_1 - a_1)]N \leq a_2 b_2$.

3 Proof of Theorem 2

We may assume that n is not a perfect square. Suppose n has more than eighteen divisors between $\sqrt{n} - \sqrt[4]{n}(\log n)^{1/7}$ and $\sqrt{n} + \sqrt[4]{n}(\log n)^{1/7}$. Say

$$n = (N - a_1)(N + b_1) = (N - a_2)(N + b_2) = \dots = (N - a_{10})(N + b_{10}) \quad (2)$$

with $0 \leq a_1 < a_2 < \dots < a_{10} \leq \sqrt[4]{n}(\log n)^{1/14}$, $0 \leq b_1 < b_2 < \dots < b_{10} \leq \sqrt[4]{n}(\log n)^{1/14}$ and $a_1 \leq b_1$ (by taking $N = \lfloor \sqrt{n} \rfloor$). By Lemma 1, we have $a_i b_i \geq N \gg n^{1/2}$ for $2 \leq i \leq 10$. Hence

$$a_i, b_i \gg n^{1/4}/(\log n)^{1/14} \quad (3)$$

By letting $l_i = b_i - a_i$, we have, from (2),

$$l_1 N - a_1(a_1 + l_1) = l_2 N - a_2(a_2 + l_2) = \dots = l_{10} N - a_{10}(a_{10} + l_{10}). \quad (4)$$

Observe that $0 \leq l_1 < l_2 < \dots < l_{10}$ as $a_1(a_1 + l_1) < a_2(a_2 + l_2) < \dots < a_{10}(a_{10} + l_{10})$. Note that by the assumption of the theorem, we have $a_1, l_1, a_1 + l_1 \leq e^{(\log n)^{2/7}}$. Also $l_i N = a_i b_i \leq \sqrt{n}(\log n)^{1/7}$ implies $l_i \leq 2(\log n)^{1/7}$. Subtracting the equations in (4), we get

$$\begin{cases} (l_2 - l_1)N = a_2(a_2 + l_2) - a_1(a_1 + l_1) \\ (l_3 - l_1)N = a_3(a_3 + l_3) - a_1(a_1 + l_1) \\ \dots\dots\dots \\ (l_{10} - l_1)N = a_{10}(a_{10} + l_{10}) - a_1(a_1 + l_1). \end{cases} \quad (5)$$

Picking the first three of (5) and eliminating N , we have

$$\begin{cases} (l_3 - l_1)a_2(a_2 + l_2) - (l_3 - l_1)a_1(a_1 + l_1) = (l_2 - l_1)a_3(a_3 + l_3) - (l_2 - l_1)a_1(a_1 + l_1) \\ (l_4 - l_1)a_2(a_2 + l_2) - (l_4 - l_1)a_1(a_1 + l_1) = (l_2 - l_1)a_4(a_4 + l_4) - (l_2 - l_1)a_1(a_1 + l_1) \end{cases}$$

Multiplying everything by 4, completing the squares and rearranging terms, we have

$$\begin{cases} (l_3 - l_1)(2a_2 + l_2)^2 - (l_2 - l_1)(2a_3 + l_3)^2 = (l_3 - l_2)(2a_1 + l_1)^2 + (l_2 - l_1)(l_3 - l_2)(l_1 - l_3) \\ (l_4 - l_1)(2a_2 + l_2)^2 - (l_2 - l_1)(2a_4 + l_4)^2 = (l_4 - l_2)(2a_1 + l_1)^2 + (l_2 - l_1)(l_4 - l_2)(l_1 - l_4) \end{cases} \quad (6)$$

Hence

$$\begin{cases} (l_4 - l_1)(l_3 - l_1)(2a_2 + l_2)^2 - (l_4 - l_1)(l_2 - l_1)(2a_3 + l_3)^2 = [(l_3 - l_2)(2a_1 + l_1)^2 + (l_2 - l_1)(l_3 - l_2)(l_1 - l_3)](l_4 - l_1) \\ (l_4 - l_1)(l_3 - l_1)(2a_2 + l_2)^2 - (l_3 - l_1)(l_2 - l_1)(2a_4 + l_4)^2 = [(l_4 - l_2)(2a_1 + l_1)^2 + (l_2 - l_1)(l_4 - l_2)(l_1 - l_4)](l_3 - l_1) \end{cases} \quad (7)$$

Let

$$(l_4 - l_1)(l_3 - l_1) = s_{34}t_{34}^2, \quad (l_4 - l_1)(l_2 - l_1) = s_{24}t_{24}^2, \quad (l_3 - l_1)(l_2 - l_1) = s_{23}t_{23}^2$$

where s_{34}, s_{24}, s_{23} are squarefree and less than $4(\log n)^{2/7}$. Then (7) becomes

$$\begin{cases} s_{34}X^2 - s_{24}Y^2 = [(l_3 - l_2)(2a_1 + l_1)^2 + (l_2 - l_1)(l_3 - l_2)(l_1 - l_3)](l_4 - l_1) \\ s_{34}X^2 - s_{23}Z^2 = [(l_4 - l_2)(2a_1 + l_1)^2 + (l_2 - l_1)(l_4 - l_2)(l_1 - l_4)](l_3 - l_1) \end{cases} \quad (8)$$

where $X = t_{34}(2a_2 + l_2)$, $Y = t_{24}(2a_3 + l_3)$ and $Z = t_{23}(2a_4 + l_4)$. To apply Theorem 5, we need to separate the exceptional cases: 1. $s_{34} = s_{24}$, 2. $s_{34} = s_{23}$, 3. $s_{34}s_{24}s_{34}s_{23}$ is a perfect square.

Case 1: $s_{34} = s_{24}$. Then

$$s_{34}(X^2 - Y^2) = [(l_3 - l_2)(2a_1 + l_1)^2 + (l_2 - l_1)(l_3 - l_2)(l_1 - l_3)](l_4 - l_1).$$

If both sides are not zero, then the absolute value of the left hand side is at least $X + Y \gg n^{1/5}$ by (3) while the right hand side is $O(e^{4(\log n)^{2/7}})$ which cannot happen if n is sufficiently large. Therefore both sides must be zero. This implies $(2a_1 + l_1)^2 = (l_3 - l_1)(l_2 - l_1)$. Recall $(l_2 - l_1)N = a_2(a_2 + l_2) - a_1(a_1 + l_1)$. Multiplying everything by 4 and completing the square, we have $(l_2 - l_1)(4N + l_2 + l_1) = (2a_2 + l_2)^2 - (2a_1 + l_1)^2$. Putting in $(2a_1 + l_1)^2 = (l_3 - l_1)(l_2 - l_1)$

and rearranging terms, we get $(l_2 - l_1)(4N + l_2 + l_3) = (2a_2 + l_2)^2$. So in this exceptional case, we have $4N + l_2 + l_3$ is a positive integer of the form ax^2 with $a \leq 2(\log n)^{1/7}$.

Case 2: $s_{34} = s_{23}$. This case is similar to case 1 and we have $4N + l_2 + l_4$ is of the form ax^2 with $a \leq 2(\log n)^{1/7}$.

Case 3: $s_{34}s_{24}s_{34}s_{23}$ is a perfect square. This implies $s_{24} = s_{23}$ since they are squarefree numbers. Subtracting the two equations in (8), we have

$$s_{23}(Z^2 - Y^2) = (l_4 - l_3)(l_1 - l_2)(2a_1 + l_1)^2 + (l_2 - l_1)(l_3 - l_1)(l_4 - l_1)(l_4 - l_3).$$

If both sides are not zero, then the absolute value of the left hand side is at least $Z + Y \gg n^{1/5}$ by (3) while the right hand side is $O(e^{4(\log n)^{2/7}})$ which cannot happen if n is sufficiently large. Therefore both sides must be zero. This implies $(2a_1 + l_1)^2 = (l_4 - l_1)(l_3 - l_1)$. Recall $(l_3 - l_1)N = a_3(a_3 + l_3) - a_1(a_1 + l_1)$. Multiplying everything by 4 and completing the square, we have $(l_3 - l_1)(4N + l_3 + l_1) = (2a_3 + l_3)^2 - (2a_1 + l_1)^2$. Putting in $(2a_1 + l_1)^2 = (l_4 - l_1)(l_3 - l_1)$ and rearranging terms, we get $(l_3 - l_1)(4N + l_3 + l_4) = (2a_3 + l_3)^2$. So in this exceptional case, we have $4N + l_3 + l_4$ is a positive integer of the form ax^2 with $a \leq 2(\log n)^{1/7}$.

Therefore, aside from these exceptions, we can apply Theorem 5 and obtain

$$n^{1/4} \ll X < e^{8C(\log n)^{6/7}(\log \log n)^5}$$

which cannot be true when n is sufficiently large. Hence we have a contradiction or one of the exceptions happens. Similarly we can pick the fourth, fifth and sixth equations in (5), argue in the same way and get a contradiction or $4N + l_i + l_j$ of the form ax^2 with $a \leq 2(\log n)^{1/7}$ for some $5 \leq i < j \leq 7$. Again we can pick the seventh, eighth and ninth equations in (5) and obtain a contradiction or $4N + l_p + l_q$ of the form ax^2 with $a \leq 2(\log n)^{1/7}$ for some $8 \leq i < j \leq 10$.

If no contradiction is obtained so far, we have three distinct positive integers $4N + l_a + l_b$, $4N + l_i + l_j$, $4N + l_p + l_q$ of the form ax^2 with $a \leq 2(\log n)^{1/7}$ for some $2 \leq a < b \leq 4$, $5 \leq i < j \leq 7$ and $8 \leq p < q \leq 10$. This fits the situation of Theorem 6. Applying it with $H = 2(\log n)^{1/7}$ and $K = 4(\log n)^{1/7}$, we get

$$(\log n)^{2/7}(\log \log n)^3((\log n)^{1/7} \log \log n)(\log \log n) \gg \log 4N \gg \log n$$

which is impossible if n is sufficiently large. With this final contradiction, we prove that n cannot have more than eighteen divisors between $\sqrt{n} - \sqrt[4]{n}(\log n)^{1/7}$ and $\sqrt{n} + \sqrt[4]{n}(\log n)^{1/7}$ and hence Theorem 2.

4 Proof of Theorem 3

Suppose

$$\#\{(a, b) : a, b \text{ integers}, a^2 + b^2 = n = N^2, |b| < n^{1/4}(\log n)^{1/7}\} > 10.$$

Then $N^2 = (N - u_1)^2 + v_1^2 = (N - u_2)^2 + v_2^2 = (N - u_3)^2 + v_3^2$ for some integers $0 < u_1 < u_2 < u_3$ and $0 < v_1 < v_2 < v_3 < n^{1/4}(\log n)^{1/7}$. $N^2 = (N - u_i)^2 + v_i^2$ implies $(N - u_i)^2 > N^2/2$ and hence $u_i < N/2$. It also gives $2Nu_i = u_i^2 + v_i^2$ and $(2N - u_i)u_i = v_i^2$. These imply $Nu_i < n^{1/2}(\log n)^{2/7}$ and $u_i < (\log n)^{2/7}$. Also $v_i^2 > N$ and $v_i > \sqrt{N} = n^{1/4}$. Now write $u_i = s_i t_i^2$ where s_i is squarefree. From above, we have $u_i = s_i t_i^2$ divides v_i^2 . This implies that $s_i t_i$ divides v_i . Write $v_i = s_i t_i w_i$, we have

$$2Ns_i t_i^2 = (s_i t_i^2)^2 + (s_i t_i w_i)^2 \text{ or } 2N = s_i t_i^2 + s_i w_i^2 \text{ for } i = 1, 2, 3.$$

Note $s_i \leq s_i t_i^2 < (\log n)^{2/7}$ and $w_i = v_i/(s_i t_i) > n^{1/4}/(\log n)^{2/7}$. Hence

$$\begin{cases} s_1 w_1^2 - s_2 w_2^2 = s_2 t_2^2 - s_1 t_1^2 \\ s_1 w_1^2 - s_3 w_3^2 = s_3 t_3^2 - s_1 t_1^2 \end{cases} \quad (9)$$

If $s_1 = s_2$, then $w_1^2 - w_2^2 = t_2^2 - t_1^2$. If the left hand side is nonzero, then its absolute value is at least $w_1 + w_2 > n^{1/4}/(\log n)^{2/7}$. However the absolute value of the right hand side is at most $(\log n)^{4/7}$. So we must have $w_1 = w_2$ and $t_1 = t_2$ which forces $u_1 = u_2$ contradicting $u_1 < u_2$. Similarly $s_1 = s_3$ is also impossible. The other exception in applying Theorem 5 is when $s_1(s_3 t_3^2 - s_1 t_1^2) = s_1(s_2 t_2^2 - s_1 t_1^2)$ which implies $u_3 = s_3 t_3^2 = s_2 t_2^2 = u_2$ contradicting $u_2 < u_3$.

Therefore we can apply Theorem 5 and obtain

$$\frac{n^{1/4}}{(\log n)^{2/7}} < w_1 < e^{C(\log n)^{4/7}(\log \log n)^3(\log n)^{2/7} \log \log n}$$

which is impossible when n is sufficiently large. Consequently we cannot have more than ten pairs of (a, b) and hence Theorem 3.

5 Proof of Theorem 4

Suppose

$$\#\{(a, b) : a, b \text{ integers}, a^2 + b^2 = n, |b| < n^{1/4}(\log n)^{1/14}\} > 36.$$

Then we have $n = a_1^2 + b_1^2 = a_2^2 + b_2^2 = \dots = a_{10}^2 + b_{10}^2$ with $a_1 > a_2 > \dots > a_{10} > 0$, $0 < b_1 < b_2 < \dots < b_{10} < n^{1/4}(\log n)^{1/14}$ and $b_1 < e^{(\log n)^{2/7}}$. Let $a_i = a_1 - l_i$ for $i = 2, 3, \dots, 10$. From $a_1^2 + b_1^2 = (a_1 - l_i)^2 + b_i^2$, we have

$$0 < 2a_1 l_i = b_i^2 - b_1^2 - l_i^2. \quad (10)$$

As $a_1 > n^{1/2}/2$ and $b_i < n^{1/4}(\log n)^{1/14}$, we have $l_i < (\log n)^{1/7}$. By eliminating the terms involving a_1 in (10), we have

$$\begin{cases} l_3 b_2^2 - l_2 b_3^2 = (l_3 - l_2)(b_1^2 - l_2 l_3) \\ l_4 b_2^2 - l_2 b_4^2 = (l_4 - l_2)(b_1^2 - l_2 l_4). \end{cases}$$

Hence

$$\begin{cases} l_3 l_4 b_2^2 - l_2 l_4 b_3^2 = l_4(l_3 - l_2)(b_1^2 - l_2 l_3) \\ l_3 l_4 b_2^2 - l_2 l_3 b_4^2 = l_3(l_4 - l_2)(b_1^2 - l_2 l_4). \end{cases}$$

Let $l_3 l_4 = s_{34} t_{34}^2$, $l_2 l_4 = s_{24} t_{24}^2$ and $l_2 l_3 = s_{23} t_{23}^2$ where s_{34} , s_{24} and s_{23} are squarefree and less than $(\log n)^{2/7}$. Therefore

$$\begin{cases} s_{34} X^2 - s_{24} Y^2 = l_4(l_3 - l_2)(b_1^2 - l_2 l_3) \\ s_{34} X^2 - s_{23} Z^2 = l_3(l_4 - l_2)(b_1^2 - l_2 l_4) \end{cases} \quad (11)$$

where $X = t_{34} b_2$, $Y = t_{24} b_3$ and $Z = t_{23} b_4$. From (10), we have $b_i^2 > 2a_1 > n^{1/2}$. Hence $X, Y, Z > n^{1/4}$. Similar to the proof of Theorem 2, we look at the exceptions in applying Theorem 5. If $s_{34} = s_{24}$, then

$$s_{34}(X^2 - Y^2) = l_4(l_3 - l_2)(b_1^2 - l_2 l_3).$$

If the left hand side is nonzero, then its absolute value is at least $X + Y > n^{1/4}$. However the right hand side is at most $e^{4(\log n)^{2/7}}$ which is a contradiction. So we must have the two sides are zero and $b_1^2 = l_2 l_3$. Putting this into (10), we have

$$2a_1 l_2 = b_2^2 - l_2 l_3 - l_2^2 \text{ or } (2a_1 + l_2 + l_3)l_2 = b_2^2.$$

So the number $2a_1 + l_2 + l_3$ is of the form ax^2 with $a < (\log n)^{1/7}$.

Similarly if $s_{34} = s_{23}$, then we have $2a_1 + l_2 + l_4$ is of the form ax^2 with $a < (\log n)^{1/7}$.

Finally if $s_{34}s_{24}s_{34}s_{23}$ is a perfect square, we must have $s_{24} = s_{23}$ as they are squarefree. Subtracting the two equations in (11), we have

$$s_{23}Z^2 - s_{24}Y^2 = l_2(l_3 - l_4)(b_1^2 - l_3 l_4).$$

Using the same argument as above, we produce $2a_1 + l_3 + l_4$ is of the form ax^2 with $a < (\log n)^{1/7}$.

Aside from these exceptions, we can apply Theorem 5 to (11) and get

$$n^{1/4} < X < e^{C(\log n)^{4/7}(\log \log n)^3(\log n)^{2/7} \log \log n}$$

which is a contradiction. Therefore we have a contradiction or one of these exceptions happen. Now we repeat the same argument for $a_1^2 + b_1^2 = a_5^2 + b_5^2 = a_6^2 + b_6^2 = a_7^2 + b_7^2$ and $a_1^2 + b_1^2 = a_8^2 + b_8^2 = a_9^2 + b_9^2 = a_{10}^2 + b_{10}^2$ and get $2a_1 + l_a + l_b$, $2a_1 + l_i + l_j$ and $2a_1 + l_p + l_q$ all of the form ax^2 with $a < (\log n)^{1/7}$ for some $2 \leq a < b \leq 4$, $5 \leq i < j \leq 7$ and $8 \leq p < q \leq 10$ if we have not already got a contradiction. Then we can apply Theorem 6 with $H < (\log n)^{1/7}$ and $K < 2(\log n)^{2/7}$, and get

$$C(\log n)^{2/7}(\log \log n)^3((\log n)^{1/7} \log \log n)(\log \log n) > \log a_1 \gg \log n.$$

This is impossible if n is sufficiently large. With this final contradiction, we prove Theorem 4.

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Department of Arts and Sciences
Victory University
255 N. Highland St.,
Memphis, TN 38111
U.S.A.
thchan@victory.edu